Generalized limit (of arbitrary discontinuous function)

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Abstract. I consider (generalized) limit of arbitrary (discontinuous) function, defined in terms of funcoids. Definition of generalized limit makes it obvious to define such things as derivative of an arbitrary function, integral of an arbitrary function, etc. It is given a definition of non-differentiable solution of a (partial) differential equation. It's raised the question how do such solutions "look like" starting a possible big future research program.

Contents

CHAPTER 1

Introduction

I defined *funcoid* and based on this generalized *limit of an arbitrary (even discontinuous) function* in [**1**].

In this article I consider generalized limits in more details.

This article is written in such a way that a reader could understand the main ideas on generalized limits without resorting to reading [**1**] beforehand, but to follow the proofs you need read that first.

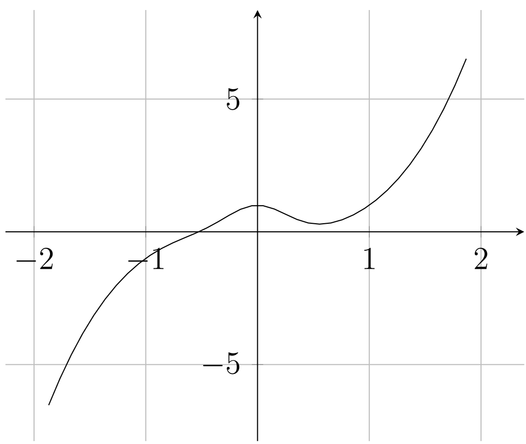
Definition of generalized limit makes it obvious to define such things as derivative of an arbitrary function, integral of an arbitrary function, etc.

Note that generalized limit is a "composite" object, not just a simple real number, point, or "regular" vector.

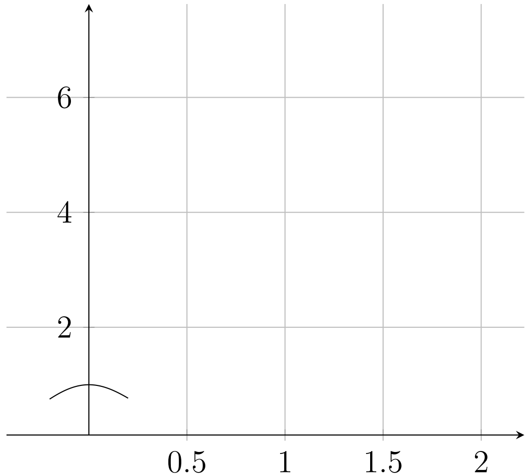
CHAPTER 2

A popular explanation of generalized limit

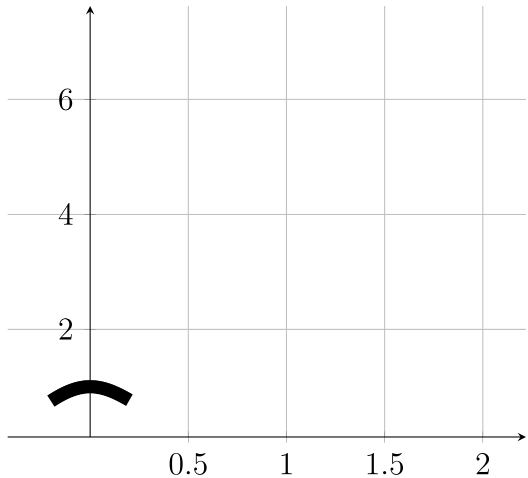
For an example, consider some real function f from x-axis to y-axis:



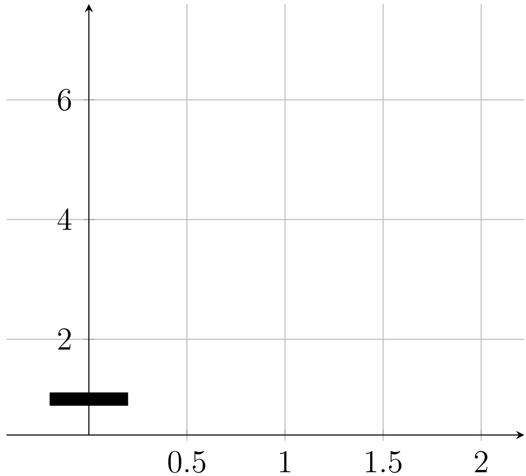
Take it's infinitely small fragment (in our example, an infinitely small interval for x around zero; see the actual book for an explanation what is infinitely small):



Next consider that with a value y replaced with an infinitely small interval like $[y - \epsilon; y + \epsilon]$:

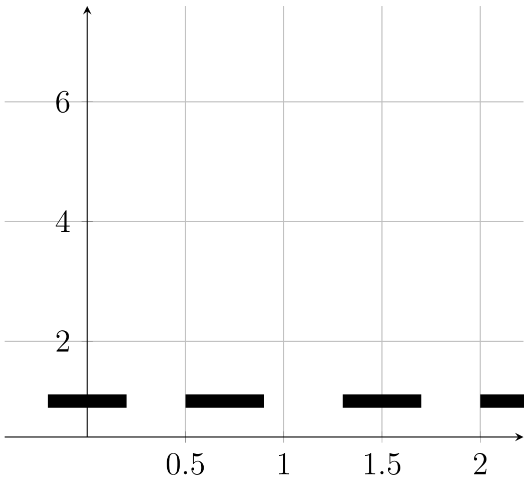


Now we have "an infinitely thin and short strip". In fact, it is the same as an "infinitely small rectangle" (Why? So infinitely small behave, it can be counter-intuitive, but if we consider the above meditations formally, we could get this result):



This infinitely small rectangle's y position uniquely characterizes the limit of our function (in our example at $x \to 0$).

If we consider the set of all rectangles we obtain by shifting this rectangle by adding an arbitrary number to x, we get



Such sets one-to-one corresponds to the value of the limit of our function (at $x \to 0$): Knowing such the set, we can calculate the limit (take its arbitrary element and get its so to say y-limit point) and knowing the limit value (y), we could write down the definition of this set.

So we have a formula for *generalized limit*:

$$\lim_{x \to a} f(x) = \{\nu \circ f|_{\Delta(a)} \circ r \mid r \in G\}$$

where G is the group of all horizontal shifts of our space $\mathbb{R}$, $f|_{\Delta(a)}$ is the function f of which we are taking limit restricted to the infinitely small interval $\Delta(a)$ around the point a, $\nu \circ$ is "stretching" our function graph into the infinitely thin “strip” by applying a topological operation to it.

What all this (especially “infinitely small”) means? It is filters and “funcoids” (see below for the definition).

Why we consider all shifts of our infinitely small rectangle? To make the limit not dependent of the point a to which x tends. Otherwise the limit would depend on the point a.

Note that for discontinuous functions elements of our set (our limit is a set) won’t be infinitely small “rectangles” (as on the pictures), but would “touch” more than just one y value.

The interesting thing here is that we can apply the above formula to *every* function: for example to a discontinuous function, Dirichlet function, unbounded function, unbounded and discontinuous at every point function, etc. In short, the generalized limit is defined for *every* function. We have a definition of limit for every function, not only a continuous function!

And it works not only for real numbers. It would work for example for any function between two topological vector spaces (a vector space with a topology).

Hurrah! Now we can define derivative and integral of *every* function.

CHAPTER 3

Funcoids

I will reprise (without proofs) several equivalent definitions of funcoid from [**1**]:

Binary relation δ between two sets (source and destination of the funcoid), conforming to the axioms:

(1) not $\emptyset \;\delta\; X$
(2) not $X \;\delta\; \emptyset$
(3) $I \cup J \;\delta\; K \iff I \;\delta\; K \wedge J \;\delta\; K$
(4) $K \;\delta\; I \cup J \iff K \;\delta\; I \wedge K \;\delta\; J$

Pair of functions (α, β) between the sets of filters filters on some two sets (source and destination of the funcoid), conforming to the formula:

$$\alpha(\mathcal{X}) \sqcap \mathcal{Y} \neq \bot \iff \beta(\mathcal{Y}) \sqcap \mathcal{X} \neq \bot.$$

REMARK 1. Funcoid (α, β) is determined by the value of α (or value of β).

A function Δ from the set of subsets of some set (source of the funcoid) to the set of filters on some set (destination of the funcoid), conforming to the axioms:

(1) $\Delta(\emptyset) = \bot$
(2) $\Delta(X \sqcup Y) = \Delta(X) \sqcup \Delta(Y)$

(Here $\sqcup$ and $\sqcap$ are the join and the meet correspondingly on the lattice of filters with order reverse to set-theoretic inclusion, $\bot$ is the improper filter.)

Note that we define things to have the equations:

(1)
$$X \,[f]^*\, Y \Leftrightarrow X \;\delta\; Y \Leftrightarrow \\ \uparrow X \,[(\alpha,\beta)]\, \uparrow Y \Leftrightarrow \alpha(X) \sqcap Y \neq \bot \Leftrightarrow \\ \beta(Y) \sqcap X \neq \bot$$

(2) $\langle f\rangle^* X = \Delta X = \alpha \uparrow X$

We will denote partial orders as $\sqsubseteq$.

I will call *endofuncoid* a funcoid whose source and destination are the same.

Funcoids form a semigroup (or precategory, dependently on the exact axioms) with the operation defined by the formula:

$$(\alpha_1,\beta_1) \circ (\alpha_0,\beta_0) = (\alpha_1 \circ \alpha_0, \beta_0 \circ \beta_1).$$

We denote $\langle(\alpha,\beta)\rangle = \alpha$ and $(\alpha,\beta)^{-1} = (\beta,\alpha)$.

Funcoids also form a poset which is a complete lattice.

Funcoids are a generalization of both topological spaces and proximity spaces (see [**1**]).

Also funcoids are ([**1**]) a generalization of binary relations. (I will denote the funcoid corresponding ([**1**]) to a binary relation f as $\uparrow f$) This makes funcoids a common generalization for topologies/proximities and functions, so they are a convenient tool to study functions between spaces.

Another important for this article operation on funcoids is *restricting* a funcoid to a filter (generalizing restricting a function to a set): $f|_{\mathcal{X}}$ for a funcoid f and filter $\mathcal{X}$.

In [**1**] we also have a funcoid called *funcoidal product* $\mathcal{X} \times^{\mathsf{FCD}} \mathcal{Y}$ of two filters $\mathcal{X}$ and $\mathcal{Y}$.

CHAPTER 4

Limit for funcoids

It is easy [**1**] to generalize topological limit for a funcoid: a funcoid f (e.g. a function) *tends* to point a ($f \to a$) regarding a funcoid ν (e.g. to a topological or proximity space) on a filter $\mathcal{X}$ iff

$$\langle f \rangle \mathcal{X} \sqsubseteq \langle \nu \rangle \uparrow \{a\}.$$

(Here $\uparrow$ denotes a principal filter corresponding to a set.)

More generally we can define: a funcoid f tends to a filter $\mathcal{A}$ ($f \to \mathcal{A}$) iff $\operatorname{im} f \sqsubseteq \mathcal{A}$ (here $\operatorname{im} f = \langle f \rangle \top = \langle f \rangle \uparrow U$ where U is the greatest set in consideration.

Then: a funcoid f tends to point a regarding a funcoid ν on a filter $\mathcal{X}$ iff $f|_{\mathcal{X}}$ tends to $\uparrow \{a\}$ regarding ν.

$\lim f$ is such a point that f tends to $\lim f$.

If ν is T_2-separable (see [**1**]), then there exists no more than one $\lim f$.

So far, not much different than limits on topological spaces (but somehow more algebraic).

CHAPTER 5

Generalized limit

1. The definition of generalized limit

In [**1**] generalized limit is defined like the formula:

$$\text{xlim}\, f = \left\{ \frac{\nu \circ f \circ \uparrow r}{r \in G} \right\}. \tag{1}$$

We suppose:

Let μ and ν be endofuncoids (on sets $\operatorname{Ob}\mu$, $\operatorname{Ob}\nu$). Let G be a transitive permutation group on $\operatorname{Ob}\mu$.

We require that μ and every $r \in G$ commute, that is

$$\mu \circ \uparrow r = \uparrow r \circ \mu. \tag{2}$$

We require for every $y \in \operatorname{Ob}\nu$

$$\nu \sqsupseteq \langle \nu \rangle \uparrow \{y\} \times^{\mathsf{FCD}} \langle \nu \rangle \uparrow \{y\}. \tag{3}$$

Proposition 2. Formula (3) follows from $\nu \sqsupseteq \nu \circ \nu^{-1}$.

Proof. Let $\nu \sqsupseteq \nu \circ \nu^{-1}$. Then

$$\begin{aligned} \langle \nu \rangle \uparrow \{y\} \times^{\mathsf{FCD}} \langle \nu \rangle \uparrow \{y\} &= \\ \nu \circ (\uparrow \{y\} \times^{\mathsf{FCD}} \uparrow \{y\}) \circ \nu^{-1} &= \\ \nu \circ \uparrow (\{y\} \times \{y\}) \circ \nu^{-1} &\sqsubseteq \\ \nu \circ 1 \circ \nu^{-1} &= \\ \nu \circ \nu^{-1} &\sqsubseteq \nu. \end{aligned}$$

(Here 1 is the identity element of the semigroup of endofuncoids.) □

Remark 3. The formula (3) usually works if ν is a proximity. It does not work if μ is a topology (or more generally pretopology or preclosure). It is however easy to turn a topology into a proximity: two sets are near if they have intersecting closures.

So we have (generalized) limits of arbitrary functions acting from $\operatorname{Ob} \mu$ to $\operatorname{Ob} \nu$. (The functions in consideration are not required to be continuous.)

Remark 4. Most typically G is the group of translations of some topological vector space[1]. So in particular we have defined limit of an arbitrary function acting from a vector topological space to a topological space.

[1] I remind that every Banach space, every normed space, and every Hilbert space is a vector topological space.

2. Injection from the set of points to the set of all generalized limits

The function τ will define an injection from the set of points of the space ν ("numbers", "points", or "vectors") to the set of all (generalized) limits (i.e. values which $\operatorname{xlim}_x f$ may take).

DEFINITION 5.

$$\tau(y) \stackrel{\text{def}}{=} \left\{ \frac{\langle\mu\rangle \uparrow \{x\} \times^{\mathsf{FCD}} \langle\nu\rangle \uparrow \{y\}}{x \in D} \right\}.$$

PROPOSITION 6.

$$\tau(y) = \left\{ \frac{(\langle\mu\rangle \uparrow \{x\} \times^{\mathsf{FCD}} \langle\nu\rangle \uparrow \{y\}) \circ \uparrow r}{r \in G} \right\}$$

for every (fixed) $x \in D$.

PROOF.

$$\begin{gathered} (\langle\mu\rangle \uparrow \{x\} \times^{\mathsf{FCD}} \langle\nu\rangle \uparrow \{y\}) \circ \uparrow r = \\ \left\langle \uparrow r^{-1} \right\rangle \langle\mu\rangle \uparrow \{x\} \times^{\mathsf{FCD}} \langle\nu\rangle \uparrow \{y\} = \\ \langle\mu\rangle \left\langle \uparrow r^{-1} \right\rangle \uparrow \{x\} \times^{\mathsf{FCD}} \langle\nu\rangle \uparrow \{y\} = \\ \langle\mu\rangle \uparrow \{r^{-1}x\} \times^{\mathsf{FCD}} \langle\nu\rangle \uparrow \{y\} \in \\ \left\{ \frac{\langle\mu\rangle \uparrow \{x\} \times^{\mathsf{FCD}} \langle\nu\rangle \uparrow \{y\}}{x \in D} \right\}. \end{gathered}$$

Reversely

$$\langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\langle\nu\rangle\uparrow\{y\} = (\langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\langle\nu\rangle\uparrow\{y\})\circ\uparrow e$$

where e is the identify element of G. □

PROPOSITION 7.

$$\tau(y) = \operatorname{xlim}(\langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\uparrow\{y\})$$

(for every x). Informally: Every $\tau(y)$ is a generalized limit of a constant function.

PROOF.

$$\operatorname{xlim}(\langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\uparrow\{y\}) = \left\{\frac{\nu\circ(\langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\uparrow\{y\})\circ\uparrow r}{r\in G}\right\} = \left\{\frac{(\langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\langle\nu\rangle\uparrow\{y\})\circ\uparrow r}{r\in G}\right\} = \tau(y).$$

□

In further we will use on of the definitions of continuity from [**1**]:

$$f\in\mathrm{C}(\mu,\nu)\Leftrightarrow f\circ\mu\sqsubseteq\nu\circ f$$

and other notation from the book.

THEOREM 8. If f is a function and $f|_{\langle\mu\rangle\uparrow\{x\}}\in\mathrm{C}(\mu,\nu)$ and $\langle\mu\rangle\uparrow\{x\}\sqsupseteq\uparrow\{x\}$ then $\operatorname{xlim}_x f = \tau(fx)$.

PROOF. $f|_{\langle\mu\rangle\uparrow\{x\}}\circ\mu\sqsubseteq\nu\circ f|_{\langle\mu\rangle\uparrow\{x\}}\sqsubseteq\nu\circ f$; thus $\langle f\rangle\langle\mu\rangle\uparrow\{x\}\sqsubseteq\langle\nu\rangle\langle f\rangle\uparrow\{x\}$; consequently we have

$$\nu\sqsupseteq\langle\nu\rangle\langle f\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\langle\nu\rangle\langle f\rangle\uparrow\{x\}\sqsupseteq \langle f\rangle\langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\langle\nu\rangle\langle f\rangle\uparrow\{x\}.$$

$$\nu\circ f|_{\langle\mu\rangle\uparrow\{x\}}\sqsupseteq (\langle f\rangle\langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\langle\nu\rangle\langle f\rangle\uparrow\{x\})\circ f|_{\langle\mu\rangle\uparrow\{x\}}= (f|_{\langle\mu\rangle\uparrow\{x\}})^{-1}\langle f\rangle\langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\langle\nu\rangle\langle f\rangle\uparrow\{x\}\sqsupseteq \left\langle\mathrm{id}^{\mathsf{FCD}}_{\operatorname{dom}f|_{\langle\mu\rangle\uparrow\{x\}}}\right\rangle\langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\langle\nu\rangle\langle f\rangle\uparrow\{x\}\sqsupseteq \operatorname{dom}f|_{\langle\mu\rangle\uparrow\{x\}}\times^{\mathsf{FCD}}\langle\nu\rangle\langle f\rangle\uparrow\{x\}= \langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\langle\nu\rangle\langle f\rangle\uparrow\{x\}.$$

$\operatorname{im}(\nu\circ f|_{\langle\mu\rangle\uparrow\{x\}})=\langle\nu\rangle\langle f\rangle\uparrow\{x\}$;

$$\nu\circ f|_{\langle\mu\rangle\uparrow\{x\}}\sqsubseteq \langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\operatorname{im}(\nu\circ f|_{\langle\mu\rangle\uparrow\{x\}})= \langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\langle\nu\rangle\langle f\rangle\uparrow\{x\}.$$

So $\nu\circ f|_{\langle\mu\rangle\uparrow\{x\}}=\langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\langle\nu\rangle\langle f\rangle\uparrow\{x\}$.

Thus

$$\operatorname{xlim}_x f= \left\{\frac{(\langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\langle\nu\rangle\langle f\rangle\uparrow\{x\})\circ\uparrow r}{r\in G}\right\}= \tau(fx).$$

□

REMARK 9. Without the requirement of $\langle\mu\rangle\uparrow\{x\}\sqsupseteq\uparrow\{x\}$ the last theorem would not work in the case of removable singularity.

THEOREM 10. Let $\nu\sqsubseteq\nu\circ\nu$. If $f|_{\langle\mu\rangle\uparrow\{x\}}\xrightarrow{\nu}\uparrow\{y\}$ then $\operatorname{xlim}_x f=\tau(y)$.

PROOF. $\operatorname{im} f|_{\langle\mu\rangle\uparrow\{x\}}\sqsubseteq\langle\nu\rangle\uparrow\{y\}$; $\langle f\rangle\langle\mu\rangle\uparrow\{x\}\sqsubseteq\langle\nu\rangle\uparrow\{y\}$;

$$\begin{aligned}\nu\circ f|_{\langle\mu\rangle\uparrow\{x\}}&\sqsupseteq\\(\langle\nu\rangle\uparrow\{y\}\times^{\mathsf{FCD}}\langle\nu\rangle\uparrow\{y\})\circ f|_{\langle\mu\rangle\uparrow\{x\}}&=\\\left\langle(f|_{\langle\mu\rangle\uparrow\{x\}})^{-1}\right\rangle\langle\nu\rangle\uparrow\{y\}\times^{\mathsf{FCD}}\langle\nu\rangle\uparrow\{y\}&=\\\left\langle\mathrm{id}^{\mathsf{FCD}}_{\langle\mu\rangle\uparrow\{x\}}\circ f^{-1}\right\rangle\langle\nu\rangle\uparrow\{y\}\times^{\mathsf{FCD}}\langle\nu\rangle\uparrow\{y\}&\sqsupseteq\\\left\langle\mathrm{id}^{\mathsf{FCD}}_{\langle\mu\rangle\uparrow\{x\}}\circ f^{-1}\right\rangle\langle f\rangle\langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\langle\nu\rangle\uparrow\{y\}&=\\\left\langle\mathrm{id}^{\mathsf{FCD}}_{\langle\mu\rangle\uparrow\{x\}}\right\rangle\left\langle f^{-1}\circ f\right\rangle\langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\langle\nu\rangle\uparrow\{y\}&\sqsupseteq\\\left\langle\mathrm{id}^{\mathsf{FCD}}_{\langle\mu\rangle\uparrow\{x\}}\right\rangle\left\langle\mathrm{id}^{\mathsf{FCD}}_{\langle\mu\rangle\uparrow\{x\}}\right\rangle\langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\langle\nu\rangle\uparrow\{y\}&=\\\langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\langle\nu\rangle\uparrow\{y\}.&\end{aligned}$$

On the other hand,

$$f|_{\langle\mu\rangle\uparrow\{x\}}\sqsubseteq\langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\langle\nu\rangle\uparrow\{y\};$$

$$\nu\circ f|_{\langle\mu\rangle\uparrow\{x\}}\sqsubseteq\langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\langle\nu\rangle\langle\nu\rangle\uparrow\{y\}\sqsubseteq\langle\mu\rangle\uparrow\{x\}\times^{\mathsf{FCD}}\langle\nu\rangle\uparrow\{y\}.$$

So $\nu \circ f|_{\langle\mu\rangle\uparrow\{x\}} = \langle\mu\rangle \uparrow \{x\} \times^{\mathsf{FCD}} \langle\nu\rangle \uparrow \{y\}$.

$$\operatorname{xlim}_x f = \left\{ \frac{\nu \circ f|_{\langle\mu\rangle\uparrow\{x\}} \circ \uparrow r}{r \in G} \right\} = \left\{ \frac{(\langle\mu\rangle \uparrow \{x\} \times^{\mathsf{FCD}} \langle\nu\rangle \uparrow \{y\}) \circ \uparrow r}{r \in G} \right\} = \tau(y).$$

□

COROLLARY 11. If $\lim^{\nu}_{\langle\mu\rangle\uparrow\{x\}} f = y$ then $\operatorname{xlim}_x f = \tau(y)$ (provided that $\nu \sqsubseteq \nu \circ \nu$).

We have injective τ if $\langle\nu\rangle \uparrow \{y_1\} \sqcap \langle\nu\rangle \uparrow \{y_2\} = \bot^{\mathscr{F}(\operatorname{Ob}\mu)}$ for every distinct $y_1, y_2 \in \operatorname{Ob}\nu$ that is if ν is T_2-separable.

3. Hausdorff and Kolmogorov funcoids

DEFINITION 12. A funcoid f is *Kolmogorov* when $\langle f\rangle \uparrow \{x\} \neq \langle f\rangle \uparrow \{y\}$ for every distinct points $x, y \in \operatorname{dom} f$.

DEFINITION 13. *Limit* $\lim \mathcal{F} = x$ of a filter $\mathcal{F}$ regarding funcoid f is such a point that $\langle f\rangle \uparrow \{x\} \sqsupseteq \mathcal{F}$.

DEFINITION 14. *Hausdorff* funcoid is such a funcoid that every proper filter on its image has at most one limit.

PROPOSITION 15. The following are pairwise equivalent for every funcoid f:

(1) f is Hausdorff.
(2) $x \neq y \Rightarrow \langle f \rangle \uparrow \{x\} \sqcap \langle f \rangle \uparrow \{y\} = \bot$.

PROOF.

1⇒2: If 2 does not hold, then there exist distinct points x and y such that $\langle f \rangle \uparrow \{x\} \sqcap \langle f \rangle \uparrow \{y\} \neq \bot$. So x and y are both limit points of $\langle f \rangle \uparrow \{x\} \sqcap \langle f \rangle \uparrow \{y\}$, and thus f is not Hausdorff.

2⇒1: Suppose $\mathcal{F}$ is proper.

$$\langle f \rangle \uparrow \{x\} \sqsupseteq \mathcal{F} \wedge \langle f \rangle \uparrow \{y\} \sqsupseteq \mathcal{F} \Rightarrow \langle f \rangle \uparrow \{x\} \sqcap \langle f \rangle \uparrow \{y\} \neq \bot \Rightarrow x = y.$$

□

COROLLARY 16. Every entirely defined Hausdorff funcoid is Kolmogorov.

REMARK 17. It is enough to be “almost entirely defined” (having nonempty value everywhere except of one point).

OBVIOUS 18. For a complete funcoid induced by a topological space this coincides with the traditional definition of a Hausdorff topological space.

CHAPTER 6

Operations on generalized limits

I will call *singularities* the set of generalized limits of the form $\operatorname{xlim}_{\langle\mu\rangle\uparrow\{x\}} f$ where f is an entirely defined funcoid and x ranges all points of $\operatorname{Ob}\mu$.

Switching back and forth between generalized limits and what I call F-singularities:

$$\Phi f = \left\{ \frac{(\operatorname{dom} F, F)}{F \in \operatorname{up} f} \right\};$$
$$\Psi f = \operatorname{im} f.$$

PROPOSITION 19. Φ is an injection from the set of singularities to the set of monovalued functions, provided the funcoid μ is Kolmogorov and ν is entirely defined.

PROOF. That it's an injection is obvious.

We need to prove that $\operatorname{dom} F_0 \neq \operatorname{dom} F_1$ for each $F_0, F_1 \in f$ such that $F_0 \neq F_1$. Really, $F_0 = \nu \circ f|_{\langle\mu\rangle\uparrow\{x_0\}} \circ \uparrow r_0$ for $x_0 \in \operatorname{Ob}\mu$, $r_0 \in G$. We have $\operatorname{dom} F_0 = \operatorname{dom} f|_{\langle\mu\rangle\uparrow\{x_0\}} = \langle\mu\rangle \uparrow \{x_0\}$. Similarly $\operatorname{dom} F_1 = \langle\mu\rangle \uparrow \{x_1\}$ for some $x_1 \in \operatorname{Ob}\mu$. Thus $\operatorname{dom} F_0 \neq \operatorname{dom} F_1$ because otherwise $x_0 = x_1$ and so $r_0 \neq r_1$,

$$\begin{aligned}\operatorname{dom} F_0 = \left\langle \uparrow r_0^{-1} \right\rangle \langle \mu \rangle \uparrow \{x_0\} = \\ \langle \mu \rangle \left\langle \uparrow r_0^{-1} \right\rangle \uparrow \{x_0\} \neq \\ \text{(Kolmogorov property)} \neq \\ \langle \mu \rangle \left\langle \uparrow r_1^{-1} \right\rangle \{x_0\} = \\ \left\langle \uparrow r_1^{-1} \right\rangle \langle \mu \rangle \{x_0\} = \operatorname{dom} F_1,\end{aligned}$$

contradiction. □

So if we define a function on the set of functions whose values are funcoids, we automatically define (as this injection preimage) a function on the set of singularities. Let's do it.

Let φ be a (possibly multivalued) multiargument function.

1. Applying functions to functions

As usually in calculus:

DEFINITION 20. $(\varphi f)x = \varphi(\lambda i \in D : f_i x)$ for an indexed family f of functions of the same domain D.

2. Applying functions to sets

DEFINITION 21. $\varphi X = \langle \varphi \rangle^* \prod X$ for a family X of sets.

OBVIOUS 22. $\varphi(\lambda i \in D : \{x_0\}) = \{\varphi x\}$.

3. Applying functions to filters

DEFINITION 23. $\varphi x = \langle\varphi\rangle \prod^{\mathsf{RLD}}_{X\in\prod x} X$ for a family x of atomic filters.

PROPOSITION 24. φ can be continued to a pointfree funcoid.

PROOF. Need to prove (theorem 1650 in [**1**])

$$\langle\varphi\rangle \prod^{\mathsf{RLD}}_{X\in\prod a} X \sqsubseteq \bigsqcap\left\{\frac{\bigsqcup\left\langle x\mapsto \langle\varphi\rangle \prod^{\mathsf{RLD}}_{X\in\prod x} X\right\rangle^{*} \operatorname{atoms}\mathcal{X}}{\mathcal{X}\in\operatorname{up} a}\right\}.$$

Really,

$$\bigsqcup\left\langle x\mapsto \langle\varphi\rangle \prod^{\mathsf{RLD}}_{X\in\prod x} X\right\rangle^{*} \operatorname{atoms}\mathcal{X} = \langle\varphi\rangle \bigsqcup\left\langle x\mapsto \prod^{\mathsf{RLD}}_{X\in\prod x} X\right\rangle^{*} \operatorname{atoms}\mathcal{X}.$$

But by theorem 1875 in [**1**]:

$$\bigsqcup\left\langle x\mapsto \prod^{\mathsf{RLD}}_{X\in\prod x} X\right\rangle^{*} \operatorname{atoms}\mathcal{X} = \prod^{\mathsf{RLD}}_{X\in\prod \mathcal{X}} X.$$

So, $\bigsqcup\left\langle x\mapsto \prod^{\mathsf{RLD}}_{X\in\prod a} X\right\rangle^{*} \operatorname{atoms}\mathcal{X} \sqsupseteq \prod^{\mathsf{RLD}}_{X\in\prod a} \mathcal{X}$. Thus follows the thesis. □

4. Applying functions to funcoids

DEFINITION 25. For a family f of funcoids and filter $\mathcal{X}$

$$(\varphi f)\mathcal{X} = \varphi\left(\lambda i \in \operatorname{dom} f : \langle f_i \rangle \mathcal{X}\right).$$

PROPOSITION 26. It is a component of a funcoid.

PROOF. As composition of two components of pointfree funcoids:
$\varphi(f_0, \ldots, f_n) = \varphi \circ (\mathcal{X} \mapsto (\lambda i \in \operatorname{dom} f : \langle f_i \rangle \mathcal{X}))$.
Note that $\mathcal{X} \mapsto (\lambda i \in \operatorname{dom} f : \langle f_i \rangle \mathcal{X})$ is a component of a pointfree funcoid because

$$\begin{aligned}
\mathcal{Y} \not\asymp (\lambda i \in \operatorname{dom} f : \langle f_i \rangle \mathcal{X}) \Leftrightarrow \\
\exists i \in \operatorname{dom} f : \mathcal{Y}_i \not\asymp \langle f_i \rangle \mathcal{X} \Leftrightarrow \\
\exists i \in \operatorname{dom} f : \mathcal{X} \not\asymp \left\langle f_i^{-1} \right\rangle \mathcal{Y}_i \Leftrightarrow \\
\mathcal{X} \not\asymp \left(\lambda i \in \operatorname{dom} f : \left\langle f_i^{-1} \right\rangle \mathcal{Y}_i\right) = \\
\mathcal{X} \not\asymp \left(\mathcal{Y} \mapsto \left(\lambda i \in \operatorname{dom} f : \left\langle f_i^{-1} \right\rangle \mathcal{Y}_i\right)\right) \mathcal{Y}.
\end{aligned}$$

□

PROPOSITION 27. Applying to funcoids is consistent with applying to functions.

PROOF. Consider values on principal atomic filters. □

5. Applying to generalized limits

DEFINITION 28. Define applying finitary (multivalued) functions φ to and indexed family x of F-singularities of the same domain D as

$$\varphi x = \lambda\Delta \in D : \langle\nu\rangle\, \varphi(\lambda i \in \operatorname{dom} x : x_i\Delta).$$

PROPOSITION 29. If ν is transitive ($\nu \circ \nu \sqsubseteq \nu$) and reflexive and ν commutes with φ in some argument k, then

$$\varphi x = \lambda\Delta \in D : \varphi(\lambda i \in \operatorname{dom} x : x_i\Delta).$$

PROOF. $\lambda\Delta \in D : \varphi(\lambda i \in \operatorname{dom} x : x_i\Delta) \sqsubseteq \varphi x$ because ν is reflexive.

$x_i = \nu \circ f_i \circ r'_i$ for some funcoid f_i and $r'_i \in G$.

$x_i\Delta = \langle\nu\rangle\,\langle f_i\rangle\,\langle r_i\rangle\,\Delta$ for some $r_i \in G$.

$$\varphi(\lambda i \in \operatorname{dom} x : x_i\Delta) = \\ \varphi(\lambda i \in \operatorname{dom} x : \langle\nu\rangle\,\langle f_i\rangle\,\langle r_i\rangle\,\Delta) \sqsupseteq \\ \varphi\left(\lambda i \in \operatorname{dom} x : \begin{cases} \langle\nu\rangle\,\langle f_i\rangle\,\langle r_i\rangle\,\Delta & \text{if } i \neq k \\ \langle\nu\rangle\,\langle\nu\rangle\,\langle f_i\rangle\,\langle r_i\rangle\,\Delta & \text{if } i = k \end{cases}\right) = \\ \langle\nu\rangle\,\varphi\left(\lambda i \in \operatorname{dom} x : \langle\nu\rangle\,\langle f_i\rangle\,\langle r_i\rangle\,\Delta\right) = \varphi x.$$

□

DEFINITION 30. Applying to singularities: $\varphi x = \Psi f(\lambda i \in \operatorname{dom} x : \Phi x_i)$ (applicable only if limits x_i are taken on filters that are equal up to $\langle r\rangle$ for $r \in G$).

THEOREM 31. If φ is continuous regarding ν in each argument and $\operatorname{dom} f_0 = \cdots = \operatorname{dom} f_n = \Delta$ and $\nu \circ \nu \sqsubseteq \nu$, then for singularities

$$\lim \varphi(f_0, \ldots, f_n) = \varphi(\lim f_0, \ldots, \lim f_n).$$

PROOF. We will prove instead

$$\varphi(\Phi \lim f_0, \ldots, \Phi \lim f_n) = \Phi \lim \varphi(f_0, \ldots, f_n).$$

Equivalently transforming:

$$\lambda\Delta \in D : \langle \nu \rangle \, \varphi((\Phi \lim f_0)\Delta, \ldots, (\Phi \lim f_n)\Delta) = \Phi \lim \varphi(f_0, \ldots, f_n);$$

$\nu \circ \varphi(\nu \circ f_0 \circ r, \ldots, \nu \circ f_n \circ r) = \nu \circ \varphi(f_0, \ldots, f_n) \circ r;$
$\nu \circ \varphi(\nu \circ f_0, \ldots, \nu \circ f_n) = \nu \circ \varphi(f_0, \ldots, f_n);$
Obviously, $\nu \circ \varphi(\nu \circ f_0, \ldots, \nu \circ f_n) \sqsupseteq \nu \circ \varphi(f_0, \ldots, f_n)$.

Reversely, applying continuity $n+1$ times, we get:

$$\nu \circ \varphi(\nu \circ f_0, \ldots, \nu \circ f_n) \sqsubseteq \underbrace{\nu \circ \nu}_{n+1 \text{ times}} \circ \varphi(f_0, \ldots, f_n) \sqsubseteq \nu \circ \varphi(f_0, \ldots, f_n).$$

So $\nu \circ \varphi(\nu \circ f_0, \ldots, \nu \circ f_n) = \nu \circ \varphi(f_0, \ldots, f_n)$. □

PROPOSITION 32. If ϕ is continuous regarding ν in each argument, then

$$\varphi(\lim f_0|_\Delta, \ldots, \lim f_n|_\Delta) = \lim \varphi(f_0|_\Delta, \ldots, f_n|_\Delta) = \lim_\Delta \varphi(f_0, \ldots, f_n)$$

for funcoids f_0, ..., f_n,

PROOF. The first equality follows from the above. It remains to prove

$$\varphi(f_0|_\Delta, \ldots, f_n|_\Delta) = (\varphi(f_0, \ldots, f_n))|_\Delta.$$

Equivalently transforming,
$\langle \varphi(f_0|_\Delta, \ldots, f_n|_\Delta)\rangle \mathcal{X} = \langle (\varphi(f_0, \ldots, f_n))|_\Delta\rangle \mathcal{X}$;

$$\varphi\left(\langle f_0|_\Delta\rangle \mathcal{X}, \ldots, \langle f_n|_\Delta\rangle \mathcal{X}\right) = \langle \varphi(f_0, \ldots, f_n)\rangle (\Delta \sqcap \mathcal{X});$$

$$\varphi\left(\langle f_0\rangle (\Delta \sqcap \mathcal{X}), \ldots, \langle f_n\rangle (\Delta \sqcap \mathcal{X})\right) = \varphi\left(\langle f_0\rangle (\Delta \sqcap \mathcal{X}), \ldots, \langle f_n\rangle (\Delta \sqcap \mathcal{X})\right) (\Delta \sqcap \mathcal{X}).$$

□

THEOREM 33. Let Δ be a filter on μ. Let S be the set of all functions $p \in \mathsf{FCD}(\operatorname{Ob}\mu, \operatorname{Ob}\nu)$ such that $\operatorname{dom} p = \Delta$. Let f, g be finitary multiargument functions on $\operatorname{Ob}\nu$. Let J be an index set. Let $k \in J^{\operatorname{dom} P}$, $l \in J^{\operatorname{dom} Q}$. Then

$$\forall x \in (\operatorname{Ob}\nu)^J : f(\lambda i \in \operatorname{dom} f : x_{k_i}) = g(\lambda i \in \operatorname{dom} g : x_{l_i})$$

implies

$$\forall x \in (\langle\lim\rangle^* S)^J : f(\lambda i \in \operatorname{dom} f : x_{k_i}) = g(\lambda i \in \operatorname{dom} g : x_{l_i}),$$

provided that f and g are continuous regarding ν in each argument.

Remark 34. This theorem implies that if $\operatorname{Ob}\nu$ is a group, ring, vector space, etc., then $\langle\lim\rangle^* S$ is also accordingly a group, ring, vector space, etc.

Proof. Every $x_{j_i} = \lim_\Delta t$ for some function t. By proved above,

$$f(\lambda i \in \operatorname{dom} f : x_{k_i}) = \lim_\Delta f(\lambda i \in \operatorname{dom} f : t_{k_i}).$$

It's enough to prove

$$f(\lambda i \in \operatorname{dom} f : t_{k_i}) = g(\lambda i \in \operatorname{dom} f : t_{l_i}).$$

But that's trivial. □

Conjecture 35. The above theorem stays true if S is instead a set of limits of monovalued funcoids.

6. Applications

Having generalized limit, we can in an obvious way define derivative of an arbitrary function.

We can also define definite integral of an arbitrary function (I remind that integral is just a limit on a certain filter). The result may differ dependently on whether we use Riemann and Lebesgue integrals.

From above it follows that my generalized derivatives and integrals are linear operators.

CHAPTER 7

Hierarchy of singularities

Above we have defined (having fixed endofuncoids μ and ν) for every set of "points" $R = \operatorname{Ob}\nu$ its set of singularities $\operatorname{SNG}(R)$.

We can further consider

$$\operatorname{SNG}(\operatorname{SNG}(R)), \operatorname{SNG}(\operatorname{SNG}(\operatorname{SNG}(R))),$$

etc.

If we try to put our generalized derivative into say the differential equation $h \circ f' = g \circ f$ on real numbers, we have a trouble: The left part belongs to the set of functions to $\operatorname{SNG}(Y)$ and the right part to the set of functions to Y, where Y is the set of solutions. How to equate them? If Y would be just $\mathbb{R}$ we would take the left part of the type $\operatorname{SNG}(\mathbb{R})$ and equate them using the injection τ defined above. But stop, it does not work: if the left part is of $\operatorname{SNG}(\mathbb{R})$ then the right part, too. So the left part would be $\operatorname{SNG}(\operatorname{SNG}(\mathbb{R}))$, etc. infinitely.

So we need to consider the entire set (*supersingularities*)

$$\mathrm{SUPER}(R) = R \cup \mathrm{SNG}(R) \cup \mathrm{SNG}(\mathrm{SNG}(R)) \cup \ldots$$

But what is the limit (and derivative) on this set? And how to perform addition, subtraction, multiplication, division, etc. on this set?

Finite functions on the set $\mathrm{SUPER}(R)$ are easy: just apply τ to arguments belonging to "lower" parts of the hierarchy of singularities a finite number of times, to make them to belong to the same singularity level (the biggest singularity level of all arguments).

Instead of generalized limit, we will use "regular" limit but on the set $\mathrm{SUPER}(R)$ (which below we will make into a funcoid) rather than on the set R.

See? We have a definition of (finite) differential equations (even partial differential equations) for discontinuous functions. It is just a differential equation on the ring $\mathrm{SUPER}(R)$ (if R is a ring).

What nondifferentiable solutions of such equations do look like? No idea! Do they contain singularities of higher levels of the above hierarchy? What about singularities in our sense at the center of a blackhole (that contain "lost" information)? We have something intriguing to research.

CHAPTER 8

Funcoid of singularities

I remind that for funcoid ν the relation $[\nu]^*$ can be thought as generalized nearness.

We will extend $[\nu]^*$ from the set R of points to the set of funcoids from a (fixed) set A to R having the same domain (or empty domain):

$$y_0 \, [\nu]^* \, y_1 \Leftrightarrow \forall x \in \operatorname{atoms} \operatorname{dom} y_0 : \langle y_0 \rangle x \, [\nu]^* \, \langle y_1 \rangle x$$

where $\operatorname{atoms} \operatorname{dom} y_0$ is the set of ultrafilters over the filter $\operatorname{dom} y_0$.

The above makes ν a pointfree funcoid (as defined in [**1**]) on this set of funcoids:

PROOF. Because funcoids are isomorphic to filters on certain boolean lattice, it's enough to prove:

$$\neg(\bot \, [\nu]^* \, y_1), \quad \neg(y_0 \, [\nu]^* \, \bot),$$
$$i \sqcup j \, [\nu]^* \, y_1 \Leftrightarrow i \, [\nu]^* \, y_1 \vee j \, [\nu]^* \, y_1,$$
$$y_0 \, [\nu]^* \, i \sqcup j \Leftrightarrow y_0 \, [\nu]^* \, i \vee y_0 \, [\nu]^* \, j.$$

The first two formulas are obvious. Let's prove the third (the fourth is similar):

$$i \sqcup j \, [\nu]^* \, y_1 \Leftrightarrow$$
$$\forall x \in \operatorname{atoms} \operatorname{dom}(i \sqcup j) : \langle y_0 \rangle \, x \, [\nu]^* \, \langle y_1 \rangle \, x \Leftrightarrow$$
$$\forall x \in \operatorname{atoms}(\operatorname{dom} i \sqcup \operatorname{dom} j) : \langle y_0 \rangle \, x \, [\nu]^* \, \langle y_1 \rangle \, x \Leftrightarrow$$
$$\forall x \in \operatorname{atoms} \operatorname{dom} i \cup \operatorname{atoms} \operatorname{dom} j :$$
$$\langle y_0 \rangle \, x \, [\nu]^* \, \langle y_1 \rangle \, x \Leftrightarrow$$
$$\forall x \in \operatorname{atoms} \operatorname{dom} i : \langle y_0 \rangle \, x \, [\nu]^* \, \langle y_1 \rangle \, x \vee$$
$$\forall x \in \operatorname{atoms} \operatorname{dom} j : \langle y_0 \rangle \, x \, [\nu]^* \, \langle y_1 \rangle \, x \Leftrightarrow$$
$$i \, [\nu]^* \, y_1 \vee j \, [\nu]^* \, y_1.$$

□

We will define two singularities being "near" in terms of F-singularities (that are essentially the same as singularities):

Two F-singularities y_0, y_1 are near iff there exist two elements of y_0 and y_1 correspondingly such that $\operatorname{dom} y_0 = \operatorname{dom} y_1$ and every $Y_0 \in y_0$, $Y_1 \in y_1$ are near.

Let's prove it defines a funcoid on the set of F-singularities:

Proof. Not $\emptyset \, [\nu]^* \, X$ and not $X \, [\nu]^* \, \emptyset$ are obvious.

It remains to prove for example

$$I \cup J \, [\nu]^* \, K \Leftrightarrow I \, [\nu]^* \, K \vee J \, [\nu]^* \, K,$$

but that's obvious. □

CHAPTER 9

Funcoid of supersingularities

It remains to define the funcoid of supersingularities.

Let y_0, y_1 be sets of supersingularities.

We will define y_0 and y_1 to be near iff there exist natural n, m such that

$$\tau^n[y_0 \cap \mathrm{SNG}^m(R)]\,[\nu]^*\,\tau^m[y_1 \cap \mathrm{SNG}^n(R)].$$

Remark 36. In this formula both the left and the right arguments of $[\nu]^*$ belong to $\mathrm{SNG}^{n+m}(R)$.

Let's prove that the above formula really defines a funcoid:

Proof. We need to show

$$\begin{aligned} \emptyset\,[\nu]^*\,y_1, \quad y_0\,[\nu]^*\,\emptyset, \\ i \cup j\,[\nu]^*\,y_1 \Leftrightarrow i\,[\nu]^*\,y_1 \vee j\,[\nu]^*\,y_1, \\ y_0\,[\nu]^*\,i \cup j \Leftrightarrow y_0\,[\nu]^*\,i \vee y_0\,[\nu]^*\,j. \end{aligned}$$

The first two formulas are obvious. Let's prove the third (the fourth is similar):

$$i \cup j \,[\nu]^*\, y_1 \Leftrightarrow$$
$$\exists n, m \in \mathbb{N} :$$
$$\tau^n[(i \cup j) \cap \mathrm{SNG}^m(R)] \,[\nu]^*\, \tau^m[y_1 \cap \mathrm{SNG}^n(R)] \Leftrightarrow$$
$$\exists n, m \in \mathbb{N} :$$
$$\tau^n[(i \cap \mathrm{SNG}^m(R)) \cup (j \cap \mathrm{SNG}^m(R)))] \,[\nu]^*$$
$$\tau^m[y_1 \cap \mathrm{SNG}^n(R)] \Leftrightarrow$$
$$\exists n, m \in \mathbb{N} :$$
$$\tau^n[i \cap \mathrm{SNG}^m(R)] \cup \tau^n[j \cap \mathrm{SNG}^m(R)))] \,[\nu]^*$$
$$\tau^m[y_1 \cap \mathrm{SNG}^n(R)] \Leftrightarrow$$
$$\exists n, m \in \mathbb{N} :$$
$$(\tau^n[i \cap \mathrm{SNG}^m(R)] \,[\nu]^*\, \tau^m[y_1 \cap \mathrm{SNG}^n(R)] \vee$$
$$\tau^n[j \cap \mathrm{SNG}^m(R)] \,[\nu]^*\, \tau^m[y_1 \cap \mathrm{SNG}^n(R)]) \Leftrightarrow$$
$$\exists n, m \in \mathbb{N} :$$
$$\tau^n[i \cap \mathrm{SNG}^m(R)] \,[\nu]^*\, \tau^m[y_1 \cap \mathrm{SNG}^n(R)] \vee$$
$$\exists n, m \in \mathbb{N} :$$
$$\tau^n[j \cap \mathrm{SNG}^m(R)] \,[\nu]^*\, \tau^m[y_1 \cap \mathrm{SNG}^n(R)] \Leftrightarrow$$
$$i \,[\nu]^*\, y_1 \vee j \,[\nu]^*\, y_1.$$

□

CHAPTER 10

Example differential equation

DEFINITION 37. I will call a function f *pseudocontinuous* on D when

$$\forall a \in D : \operatorname{xlim}_{\Delta\{a\}\setminus\{a\}} f = f(a).$$

1. Arbitrary pseudocontinuous continuations

Note that arbitrary pseudocontinuous continuations of generalized solutions of differential equations (diffeqs) are silly:

Let $A(f(x), f'(x)) = 0$ is a diffeq and let the equality is undefined at some point (e.g. contains division by zero). Let f be its solution with derivative f'. Replace the value in undefined point x of the solution by an arbitrary value y and calculate the derivative y' at this point. No need to hold $A(y, y') = 0$ at this point because the point is outside of the domain of the original solution. Then replace in our solution f the value at this point x by y and the derivative by y'. Then we have another continuation of the solution because the equality $A(f(x), f'(x)) = 0$ holds both for the point x and all other points.

Thus, we can take any solution and add one point of it with an arbitrary value. That's largely a nonsense from the practical point of view. (Why we would arbitrarily change one point of the solution?)

2. Solutions with pseudocontinuous derivative

So I will require for generalized solutions instead that the derivative f' is pseudocontinuous.

Next, we will consider a particular example, the diffeq $y'(x) = -1/x^2$. Let us find its continuations of generalized solutions y to the entire real line (including $x = 0$) with a y' being pseudocontinuous.

As it's well known, its solutions in the traditional sense are $y(x) = c_1 + \frac{1}{x}$ for $x < 0$ and $y(x) = c_2 + \frac{1}{x}$ for $x > 0$ where c_1, c_2 are arbitrary constants. The derivative is $y'(x) = -1/x^2$.

Remark 38. We could consider solutions on the space of supersingularities and it would be the same, except that we would be allowed to take c_1, c_2 arbitrary supersingularities instead of real numbers. This is because the supersingularities form a ring and thus the algorithm of solving the diffeq is the same as for the real numbers, thus producing the solutions of the same form.

Let's find the pseudocontinuous generalized derivative at zero by pseudocontinuity:

$$y'(0) = \lim_{x \to 0} \frac{1}{x^2}.$$

On the other hand, by the definition of derivative

$$y'(0) = \lim_{\varepsilon \to 0} \frac{c_i + \frac{1}{\varepsilon} - f(0)}{\varepsilon}.$$

The equality is possible only when $c_1 = c_2 = c = f(0)$.

So, finally, our solution is $y(x) = c + \frac{1}{x}$ for $x \neq 0$ and $y(0) = c$.

A thing to notice that now the solution is "whole": it exists at zero and does not split to two "branches" with independent constants. Our $y(x)$ is a real function, but the derivative has a singularity in my sense.

We considered generalized solutions with pseudocontinuous derivative. It is apparently the right way to define a class of generalized solutions. Now I will consider also several apparently wrong classes of solutions.

3. Pseudocontinuous generalized solutions

Let us try to require the solution of our diffeq to be pseudocontinuous instead of its derivative to be pseudocontinuous.

We already have the solution for nonzero points. For zero:

$$y'(0) = \lim_{x\to 0}(-1/x^2).$$

So the derivative:

$$y'(0) = \lim_{x\to 0}\frac{y(x)-y(0)}{x} = \lim_{x\to 0}\frac{y(x)}{x} - \lim_{x\to 0}\frac{y(0)}{x} = \lim_{x\to 0}\frac{1}{x^2} - \lim_{x\to 0}\frac{y(0)}{x}.$$

The equality is impossible.

Bibliography

[1] Victor Porton. Algebraic general topology. volume 1. edition 1. At https://mathematics21.org/algebraic-general-topology-and-math-synthesis/, 2019.